Minimum Linear Ordering with Submodular/Supermodular Functions

Satoru Iwata
(RIMS, Kyoto University)
Prasad Tetali Pushkar Tripathi
(Georgia Institute of Technology)

Submodular Optimization

- Linear Function
  - Submodular Load Balancing (SF08)
  - Submodular Vertex Cover (GKTW09, IN09)
- Cut Capacity Function
  - Symmetric Submodular Minimization (Q98)
  - Submodular Partition (ZNI05)
  - Submodular Sparsest Cut (SF08)
  - Submodular Multiway Partition (CE11)
- Coverage Function
  - Supermodular/Submodular Linear Ordering

Min Sum Set Cover

Feige, Lovász, Tetali (2004)

Hypergraph $H=(V,E)$

Linear Ordering $\sigma : V \to \{1,\dots,n\}$

$\hat{\sigma}(e) := \min\{\sigma(v) \mid v \in e\} \quad (e \in E)$

Find a linear ordering σ minimizing $\sum_{e \in E} \hat{\sigma}(e)$

Min Sum Set Cover

Feige, Lovász, Tetali (2004)

Find a linear ordering σ minimizing $\sum_{e \in E} \hat{\sigma}(e)$

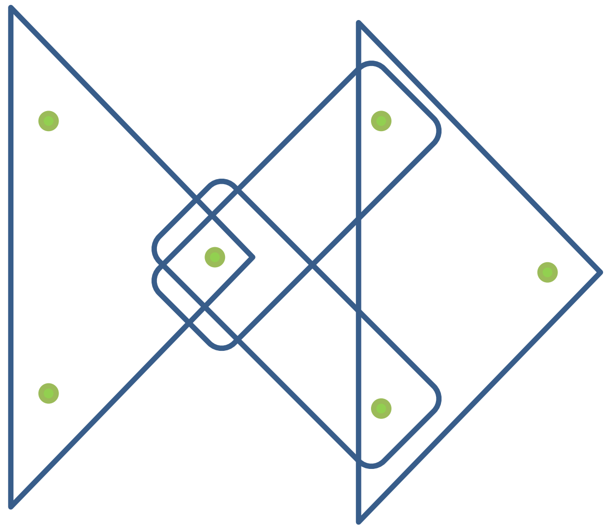
Greedy Algorithm

Repeat

$$v := \arg\max_{u \in V} \deg(u),$$

Delete v from H,

Until $V = \phi$.



Min Sum Set Cover

Feige, Lovász, Tetali (2004)

Find a linear ordering σ minimizing $\sum_{e \in E} \hat{\sigma}(e)$

- The greedy algorithm approximates MSSC within a ratio of 4.
- NP-hard to approximate MSSC within a ratio better than 4.

Min Sum Vertex Cover

Feige, Lovász, Tetali (2004)

Graph $G=(V,E)$

- An LP-rounding algorithm approximates MSVC within a ratio of 2.
- There exists a constant $\rho > 1$ such that it is NP-hard to approximates MSVC within a ratio better than ρ.

$\longrightarrow$ A 1.79-approximation algorithm based on multi-stage LP-rounding.

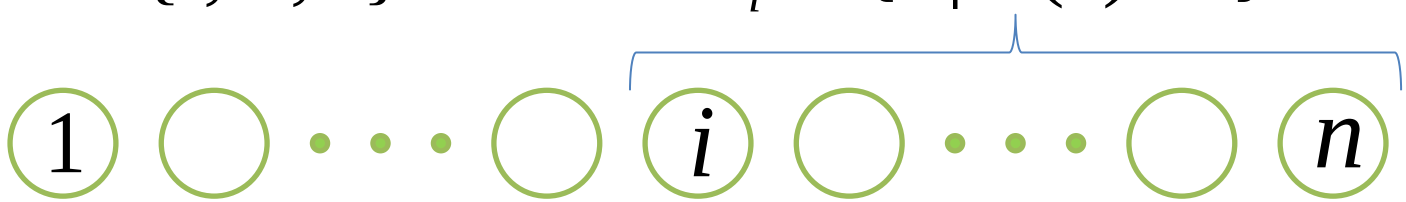
Supermodular Linear Ordering

$f : 2^V \to \mathbf{R}_+^V$ Supermodular $\qquad f(\phi) = 0$

$$f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y), \quad \forall X, Y \subseteq V.$$

Linear Ordering

$\sigma : V \to \{1, \ldots, n\}$ $\qquad S_i := \{v \mid \sigma(v) \geq i\}$



Find a linear ordering σ minimizing $\sum_{i=1}^{n} f(S_i)$

Supermodular Linear Ordering

MSSC is a special case.

$f(X)$: Number of edges included in $X \subseteq V$.

$\longrightarrow$ f : Supermodular $\quad f(\phi) = 0.$

$$e \subseteq S_i \Leftrightarrow \hat{\sigma}(e) \geq i$$

$$\boxed{\sum_{i=1}^{n} f(S_i) = \sum_{e \in E} \hat{\sigma}(e)}$$

1 $\quad$ $\cdots$ $\quad$ i $\quad$ $\cdots$ $\quad$ n

Supermodular Linear Ordering

Greedy Algorithm $\longrightarrow$ 4-Approximate Solution

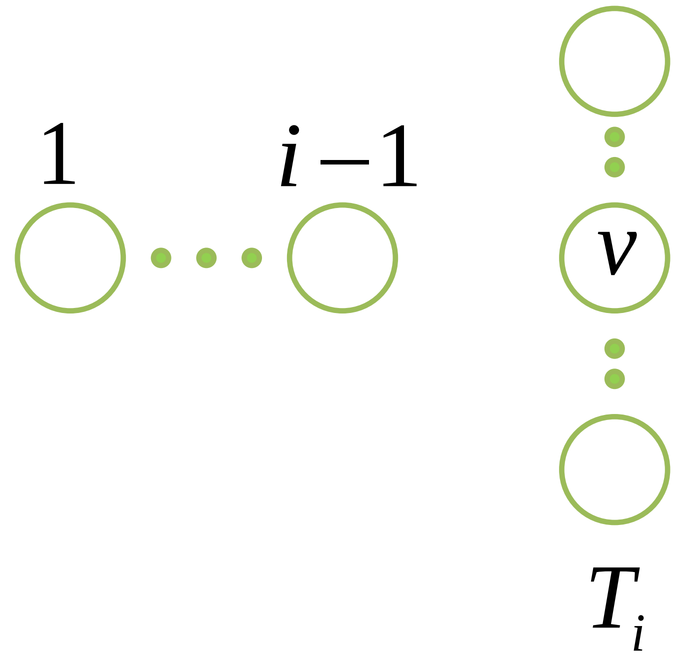
$$T_1 := V.$$

For $i = 1, \ldots, n$ do:

$$v := \arg\min_{u \in V} f(T_i \setminus \{u\}),$$

$$\sigma(v) := i,$$

$$T_{i+1} := T_i \setminus \{v\}.$$

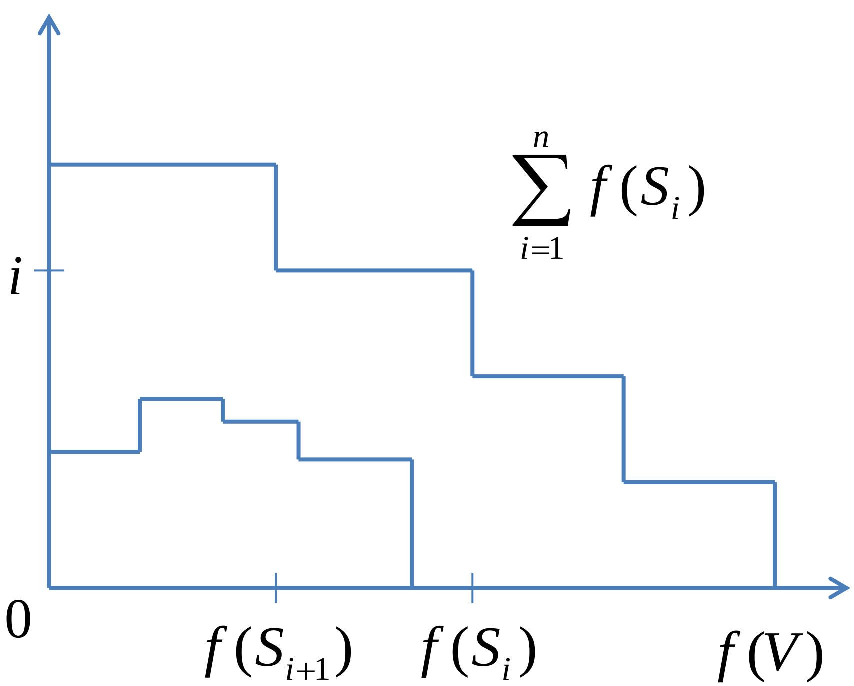


$$V = T_1 \supset T_2 \supset \cdots \supset T_n \supset T_{n+1} = \phi.$$

Supermodular Linear Ordering

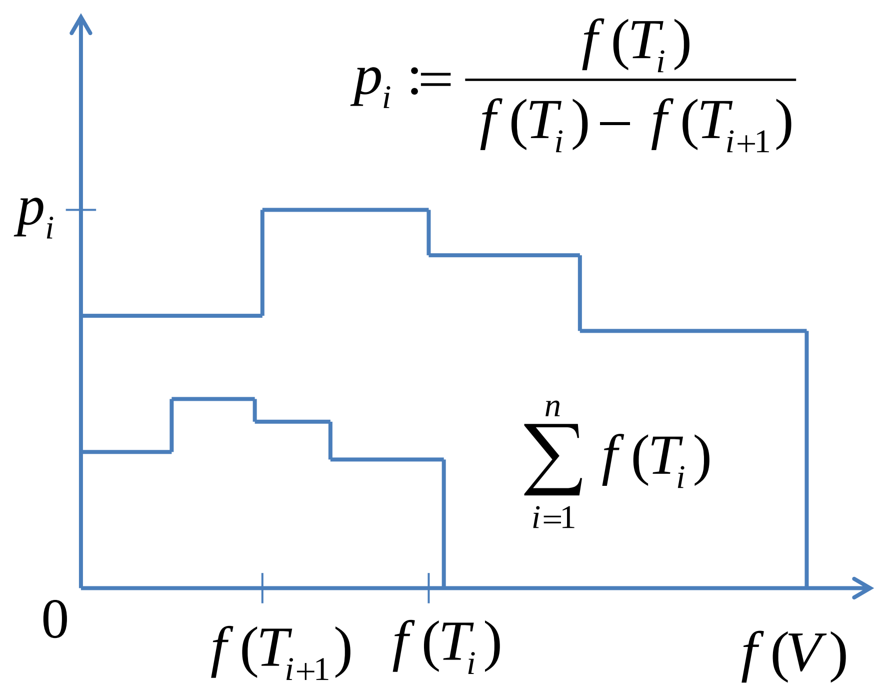
Optimal Solution

$$S_1 \supset S_2 \supset \cdots \supset S_n$$



Greedy Solution

$$T_1 \supset T_2 \supset \cdots \supset T_n$$



Supermodular Linear Ordering

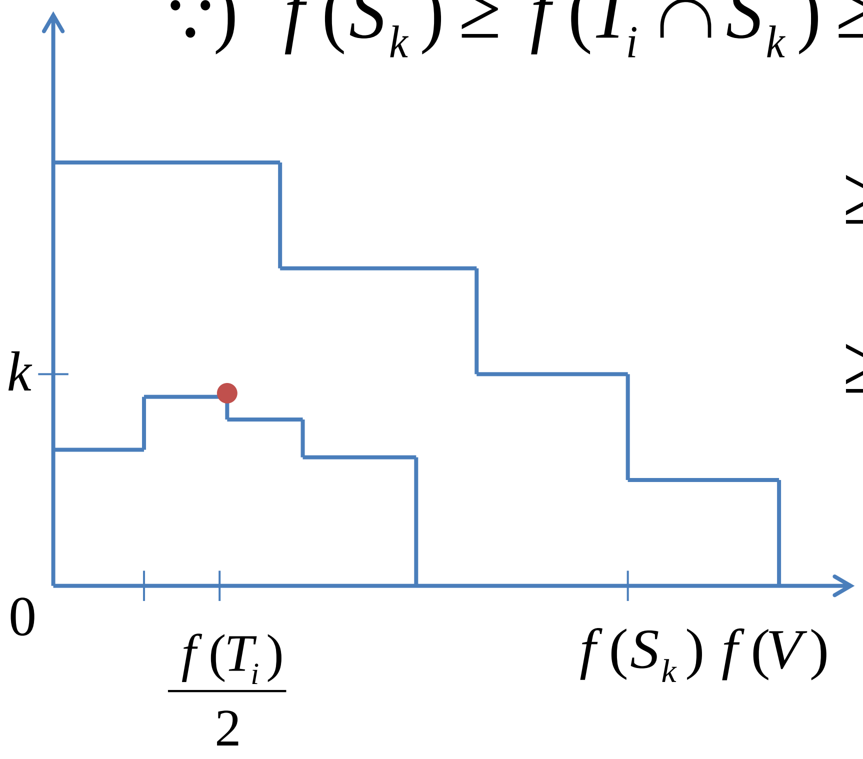
$$k := \left\lceil \frac{p_i}{2} \right\rceil \qquad \boxed{f(S_k) \ge \frac{f(T_i)}{2}}$$

$$\because)\ f(S_k) \ge f(T_i \cap S_k) \ge f(T_i) - \sum_{u \in T_i \setminus S_k} \left[f(T_i) - f(T_i \setminus \{u\}) \right]$$

$$\ge f(T_i) - (k-1)\left[f(T_i) - f(T_{i+1}) \right]$$

$$\ge f(T_i) - \frac{p_i}{2}\left[f(T_i) - f(T_{i+1}) \right]$$

$$= \frac{f(T_i)}{2}$$



Multiple Intents Reranking

Azar, Gamzu, Yin (2009)

Hypergraph $H = (V, E)$

Weight Vector $w(e) = \left\langle w_1(e), w_2(e), \ldots, w_{|e|}(e) \right\rangle \quad (e \in E)$

Linear Ordering $\sigma : V \to \{1, \ldots, n\}$

$\hat{\sigma}_j(e)$: jth smallest $\sigma(v)$ among $v \in e$.

Find a linear ordering σ that minimizes $\sum_{e \in E} \sum_{j=1}^{|e|} w_j(e) \hat{\sigma}_j(e)$.

Multiple Intents Reranking

Azar, Gamzu, Yin (2009)

- An $O(\log r)$-approximation algorithm $(r = \max_{e \in E} |e|)$.
- For non-increasing weight vector,

  a greedy 4-approx. algorithm.
- For non-decreasing weight vector,

  an LP-rounding 2-approx. algorithm.

- Constant factor approximation algorithm

  Bansal, Gupta, Krishnaswamy (2010),

  Skutella & Williamson (2011).

Multiple Intents Reranking

Hypergraph $H = (V, E)$

Weight Vector $w(e) = \left\langle w_1(e), w_2(e), \ldots, w_{|e|}(e) \right\rangle \quad (e \in E)$

$$f(X) := \sum_{e \in E} f_e(X), \qquad f_e(X) := \sum_{j > |e \setminus X|} w_j(e).$$

$$\boxed{\sum_{i=1}^{n} f(S_i) = \sum_{e \in E} \sum_{j=1}^{|e|} w_j(e) \hat{\sigma}_j(e).}$$

$w(e)$: Nonincreasing $\Rightarrow$ f_e : Supermodular

$w(e)$: Nondecreasing $\Rightarrow$ f_e : Submodular

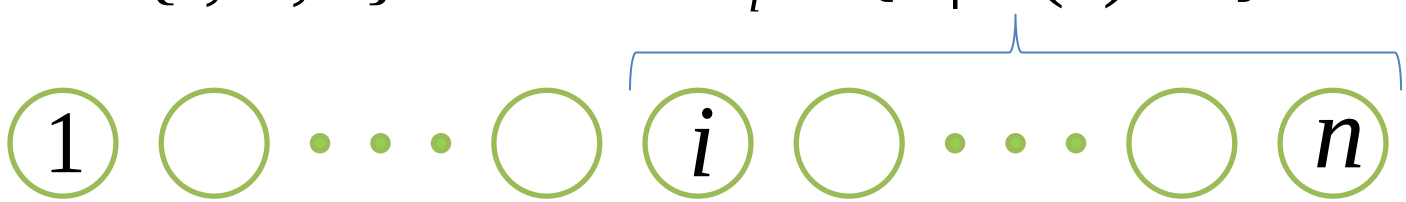
Submodular Linear Ordering

$f : 2^V \to \mathbf{R}_+^V$ Submodular $\qquad f(\phi) = 0$

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y), \quad \forall X, Y \subseteq V.$$

Linear Ordering

$\sigma : V \to \{1, \ldots, n\} \qquad S_i := \{v \mid \sigma(v) \geq i\}$



Find a linear ordering σ minimizing $\sum_{i=1}^{n} f(S_i)$

Convex Extension

Lovász (1983)

$$f : 2^V \to \mathbf{R} \qquad f(\phi) = 0$$

$$\hat{f} : \mathbf{R}_+^V \to \mathbf{R}$$

$$\hat{f}(x) := \sum_{i=1}^{n} \lambda_i f(S_i)$$

$$V = S_1 \supset S_2 \supset \cdots \supset S_n \supset \phi$$

$$x = \sum_{i=1} \lambda_i \chi_{S_i}, \quad \lambda_i \ge 0 \quad (i = 1, \ldots, n)$$

f : Submodular

$\Updownarrow$

$\hat{f}$: Convex

Submodular Linear Ordering

Convex Programming Relaxation

Minimize $\hat{f}(x)$

subject to $\displaystyle\sum_{v \in S} x(v) \geq \frac{|S|(|S|+1)}{2}, \qquad \forall S \subseteq V.$

Linear Ordering $\sigma : V \to \{1, \ldots, n\}$

$$x^{\sigma}(v) := \sigma(v)$$

$\longrightarrow$ x^{σ} : Feasible, $\displaystyle\hat{f}(x^{\sigma}) = \sum_{i=1}^{n} f(S_i).$

Submodular Linear Ordering

x^* : Optimal to the Convex Programming Relaxation

σ : Linear Ordering s.t.

$$x^*(u) \le x^*(v) \Rightarrow \sigma(u) \le \sigma(v).$$

$$\boxed{\sigma(v) \le \left(2 - \frac{2}{\sigma(v)+1}\right) x^*(v), \quad \forall v \in V.}$$

$\because$) $S := \{u \mid \sigma(u) \le \sigma(v)\}$

$$\sum_{u \in S} x^*(u) \ge \frac{\sigma(v)(\sigma(v)+1)}{2}, \qquad x^*(v) \ge \frac{\sigma(v)+1}{2}.$$

Submodular Linear Ordering

f : Monotone Submodular

$\Rightarrow \ \sigma : \left(2-\dfrac{2}{n+1}\right)$-Approximate Solution

$\because)$

$$x^{\sigma}(v) \leq \left(2-\frac{2}{\sigma(v)+1}\right)x^{*}(v) \leq \left(2-\frac{2}{n+1}\right)x^{*}(v),$$

$$\sum_{i=1}^{n} f(S_i) = \hat{f}(x^{\sigma}) \leq \left(2-\frac{2}{n+1}\right)\hat{f}(x^{*}).$$

Minimum Linear Arrangement

Graph $G=(V,E)$ Capacity $c:E\rightarrow\mathbf{R}_+$

Find a linear ordering σ minimizing

$$\sum_{(u,v)\in E} c(u,v)\,|\,\sigma(u)-\sigma(v)\,|$$

κ : Cut Capacity Function $S_i := \{v \mid \sigma(v)\geq i\}$

$$\sum_{i=1}^{n}\kappa(S_i)=\sum_{i=1}^{n}\sum\{c(u,v)\mid (u,v)\in E, u\in S_i, v\notin S_i\}$$

$$=\sum_{(u,v)\in E} c(u,v)\,|\,\sigma(u)-\sigma(v)\,|.$$

Minimum Linear Arrangement

- An $O(\log n)$ factor algorithm

  Rao & Richa (1998)

- An $O(\sqrt{\log n}\log\log n)$ factor algorithm

  Feige & Lee (2007)

- Hard to approximate within a factor of $O(\log\log n)$ under UGC.

  Devanur, Khot, Saket, Vishnoi (2006)

Submodular Linear Ordering

Unconditional Lower Bound Svitkina (2011)

No polynomial algorithms can achieve a ratio better than 2 for symmetric submodular functions.

$$f_1(S) = \min\left\{|S|, \frac{n}{2}\right\} - \frac{|S|}{2},$$

$$f_2(S) = \min\left\{|S|, \frac{n}{2}, \beta + |S \cap R|, \beta + |S \setminus R|\right\} - \frac{|S|}{2}$$

Open Problems

- An Approximation Algorithm for General (Symmetric) Submodular Linear Ordering?

- A Common Generalization of Supermodular/Submodular Linear Ordering and Multiple Intents Reranking?